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A fluid film squeezed between two parallel plane surfaces

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We study the motion which results when a fluid film is squeezed between two parallel plane surfaces in relative motion. Particular attention is given to the special case where one surface is fixed and the other is rapidly accelerated from a state of rest to a state of uniform motion. The analysis is based in part on linear theory and in substance on a finite-difference analysis of the full nonlinear equations of motion.

1. Introduction

The fluid dynamics of a thin film of lubricant which is squeezed between two parallel plane surfaces has for long been recognized to be one of the most basic problems in lubrication theory. The earliest attempts at the problem can be traced to Stefan (1874) and to Reynolds (1886), both of whom confined their attention to the special case where inertial forces are negligible in comparison to viscous forces.

The role played by inertia was first discussed by Jackson (1962); his error in the approximation of the inertia terms was repeated in the work of Kuzma, Maki & Donnelly (1964). Kuzma (1967) presented experimental results which compared favourably with the results of a correct first-order regular perturbation analysis in which the expansion parameter was the Reynolds number. Examination of the experimental results indicates, however, that far from being small the Reynolds numbers in the experiments range up to at least 60 (at which value the solution which is being perturbed, i.e. the classical lubrication approximation, is far from satisfactory). Later experimental evidence, relevant to the case of a stationary lower surface and an upper surface which is rapidly accelerated to a state of uniform motion, confirms the accuracy of regular perturbation theory for Reynolds numbers up to at least 14 (Tichy & Winer 1970).

The second-order perturbation solution was included in a wide-ranging study of the squeezed-film problem by Ishizawa (1966). Ishizawa drew attention to the rapid decrease in the magnitude of the universal functions which occur in the perturbation expansion and pointed to the obvious relevance of this property (for impulsive motion of the upper surface the expansion for the Stokes stream function F is of the form

$$F = \sum_{i=0}^{n} (ReT)^i f_i,$$

where Re denotes the Reynolds number, $T = 1-t$, where t denotes non-dimensional time, and the universal functions f_i depend only on z/T where z denotes the non-dimensional distance of separation of the planes at time t). The sixth-order perturbation solution given by MacDonald (1977) verified that for i in the range $1 \leqslant i \leqslant 5$

$$|f_{i+1}/f_i| = O(10^{-2}).$$

Furthermore, the experimental results relate to the normal force on the upper surface and when this normal force is calculated by means of the regular perturbation solution it is seen that the contribution of the second-order term is roughly 0·005 (ReT) that of the first-order term. Thus for $Re = 100$ and for T not small the second- and higher-order corrections are significant.

The regular perturbation solution essentially eliminates time from the governing partial differential equations and does not satisfy the initial condition which describes the manner in which squeezing is initiated. Thus in general there will be a transition period during which the regular perturbation solution will not accurately approximate the exact solution, regardless of the value of the Reynolds number. For the special case of a fixed lower surface and an upper surface which moves impulsively from rest to a constant velocity, the length of this transition period has been discussed by Jones & Wilson (1975).

That there is a similarity solution to the governing equations in the special case where the distance of separation of the planes is proportional to $(1-\beta t)^{\frac{1}{2}}$, where β denotes an arbitrary constant, was first observed by Ishizawa (1966). His solution was re-discovered by Wang (1976). Both authors discuss the case where $\beta < 0$ and express reservations with the resulting numerical solution.

In this paper we shall investigate the case of a lower surface which is fixed and an upper surface which is rapidly accelerated from a state of rest to a state of uniform motion. In such circumstances the initial motion of the fluid is well approximated by a linear equation, whereas the subsequent motion is nonlinear and must be studied numerically. In practice, lubrication bearings normally operate at low Reynolds numbers, but for completeness we shall study flows for which the Reynolds numbers range from 0·5 to 96·0; particular attention will be given to the normal force (or load) which the fluid exerts on the (finite) upper surface and a comparison of the load as predicted by the numerical and first-order perturbation solutions will be made.

Rapid acceleration of the upper surface will result in rapid acceleration of the fluid, the driving force being the radial pressure gradient. Thus, in the early stages of the motion, there will exist thin layers of vorticity adjacent to both surfaces with an inviscid core in between. The time required for these layers to merge is of order t_m, where

$$t_m = 2/Re + 1 - 2(1+Re)^{\frac{1}{2}}/Re;$$

Re is here based on the velocity to which the upper surface is accelerated and on the initial separation of the surfaces. Near to the start of the motion and near to the end the normal force exerted on the upper surface will be large when compared with the average normal force exerted during the motion, the large value being necessary in the former case in order to overcome fluid inertia and in the latter case to overcome the viscous force exerted on the outward-moving fluid.

2. Formulation

The geometry with which we shall be concerned is that of two parallel plane surfaces which initially are separated by a film of lubricant of thickness H. We shall assume that the lower surface is fixed and that for $0 < t^* < t_c^*$ the upper surface is in motion in the direction of its inward-drawn normal (t_c^* denotes the time at which collision would

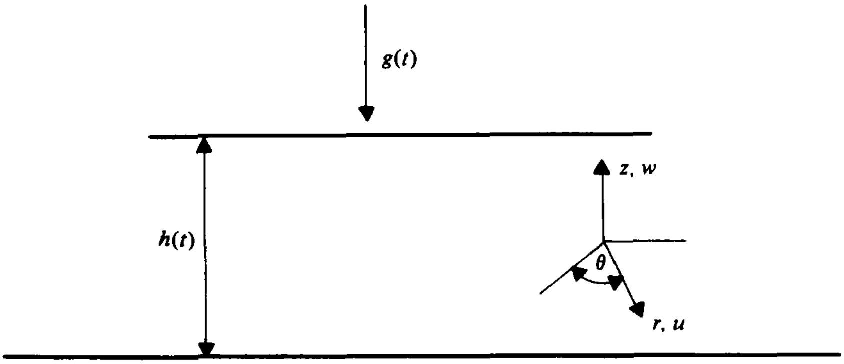

FIGURE 1. System configuration at time t (co-ordinate scheme inset).

occur). We select polar co-ordinates r^*, θ^*, z^* in terms of which the lower surface is described by $z^* = 0$ and the upper surface by $z^*/H = h(t^*V/H)$, where V denotes a representative velocity and $h(0) = 1$. We assume that the velocity of the upper surface is given by $-Vg(t^*V/H)$, where $g(\xi) > 0$, $\xi > 0$, so that its position at time t^* is described by

$$\frac{z^*}{H} = 1 - \frac{V}{H}\int_0^{t^*} g\left(\frac{\lambda V}{H}\right) d\lambda.$$

We non-dimensionalize all lengths with respect to H, all velocities with respect to V, time with respect to H/V and pressure with respect to ρV^2, where ρ denotes density. The dimensionless co-ordinates and associated velocity components we denote by r, θ, z and u, v, w respectively; t and p denote dimensionless time and pressure. The configuration is sketched in figure 1.

The mass conservation equation is satisfied by a stream function $F(z,t)$ which is such that

$$u = r\frac{\partial F}{\partial z}, \quad w = -2F,$$

so that the non-dimensional momentum equations may be expressed in the form

$$\frac{\partial^2 F}{\partial z\,\partial t} + \left(\frac{\partial F}{\partial z}\right)^2 - 2F\frac{\partial^2 F}{\partial z^2} = -\frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{Re}\frac{\partial^3 F}{\partial z^3}, \tag{2.1}$$

$$\frac{\partial F}{\partial t} - 2F\frac{\partial F}{\partial z} = \frac{1}{2}\frac{\partial p}{\partial z} + \frac{1}{Re}\frac{\partial^2 F}{\partial z^2}, \tag{2.2}$$

where $Re = VH/\nu$. Further, equations (2.1) and (2.2) show that p is of the form

$$p = \tfrac{1}{2}r^2 P_1(t) + P_2(z,t),$$

whence differentiation of (2.1) with respect to z and use of the transformation of variables

$$y = z/h(t), \quad \tau = t/Re \tag{2.3}$$

leads to the equation

$$h(Re\,\tau)\{Re\,g(Re\,\tau)[yF_{yyy} + 2F_{yy}] - 2Re\,FF_{yyy} + h(Re\,\tau)F_{yy\tau}\} = F_{yyyy}, \tag{2.4}$$

The boundary conditions under which (2.4) is to be integrated are

$$F = \frac{\partial F}{\partial y} = 0, \quad y = 0; \quad F = \tfrac{1}{2}g(t), \quad \frac{\partial F}{\partial y} = 0, \quad y = 1. \tag{2.5}$$

In the special case where $g(t) = 1$ for all $t > 0$, (2.4) and (2.5) reduce to the equations given by Jones & Wilson (1975) who considered the upper surface to move impulsively with constant velocity at time $t = 0$. Impulsive motion of the upper surface is possible only when the normal force exerted on the surface is infinite at $t = 0$, this feature being evident from the initial condition relevant to impulsive motion (Jones & Wilson 1975)

$$F = \tfrac{1}{2}y, \quad \tau = 0^+, \quad y \neq 0, 1. \tag{2.6}$$

To examine, for all $\tau > 0$, the fluid dynamics corresponding to large finite acceleration from a state of rest it is necessary to impose the initial condition

$$F = 0, \quad \tau = 0. \tag{2.6a}$$

If $F_i(y, \tau)$ and $F_r(y, \tau)$ denote, respectively, the solutions of (2.4) corresponding to impulsive and rapid accelerations then it can be shown that for $\tau > \tau_s$, where $0 < \tau_s \ll 1$, the maximum of $|F_i - F_r|$ can be made arbitrarily small by imposing sufficiently rapid acceleration. In our numerical work we shall, therefore, start the integration of (2.4) at $\tau = \tau_s$ and predict the value of F at τ_s by use of the impulsive motion solution. Further details on this matter are presented in § 4.1.

If the upper surface is assumed to be a disk of (non-dimensional) radius a and of negligible thickness the resultant normal force, or load, W, is given by

$$W = \int_0^a 2\pi r[p(r, h, t) - p_0]\, dr,$$

where $p(r, h, t)$ denotes the pressure at radial position r on the underside of the disk and p_0 denotes the pressure on the upper side. The requirement that the fluid velocity component normal to the disk is zero at the edge of the disk implies that $\partial p/\partial z$ is there equal to zero and that $p = p_0$; thus the above result may be expressed in the form

$$W = -\pi \int_0^a r^2 \frac{\partial p}{\partial r}\, dr. \tag{2.7}$$

3. Approximate analytic results

Approximate analytic results can be obtained for the special cases where

(i) $Re \ll 1$ and (ii) $0 < t \ll 1$.

Case $Re \ll 1$

This case corresponds to the regular perturbation solution referred to in § 1. The terms of the perturbation expansion can readily be derived from the equation

$$\frac{\partial^3 F}{\partial z^2 \partial t} - 2F\frac{\partial^3 F}{\partial z^3} = \frac{1}{Re}\frac{\partial^4 F}{\partial z^4}, \tag{3.1}$$

which results when p is eliminated from equations (2.1) and (2.2). The leading term is

$$F_0 = \eta^2(\tfrac{3}{2} - \eta)\, g(t), \quad \eta = z/h(t). \tag{3.2}$$

The approximation obtained when the left-hand side of (3.1) is evaluated by means of (3.2) corresponds to the first-order perturbation solution. The normal force on the upper surface as computed from this approximation shows good agreement with the experimental results over a range of Reynolds numbers extending to at least 60.

Case $0 < t \ll 1$

For an upper surface which is impulsively accelerated from a state of rest to a state of uniform motion the distance to which vorticity will have diffused in time t is $O((t/Re)^{\frac{1}{2}})$ i.e. $O(\tau^{\frac{1}{2}})$. Within this distance the change in the radial velocity profile is $O(1)$ so that F_{yy}, for example, is $O(\tau^{-\frac{1}{2}})$. An order-of-magnitude analysis performed on equation (2.4) now shows that for $Re\,\tau^{\frac{1}{2}} = o(1)$ it approximates to

$$F_{yy\tau} = F_{yyyy}, \tag{3.3}$$

which equation is to be solved subject to conditions (2.5), (2.6). With obvious qualification, the above is equally true in the case of a rapidly accelerated upper surface. Within the range in which (2.4) approximates to (3.3) it is to be expected that the regular perturbation solution will not be in good agreement with the exact solution.

For the case where $g(t) = 1$ for all $t > 0$ Jones & Wilson (1975) have used the method of separation of variables to obtain the solution for F in the form of an infinite series which is slowly convergent for $0 < \tau \ll 1$. Here, we employ the Laplace transformation to obtain a solution which is rapidly convergent for $\tau \to 0$. From (3.3), (2.6) and (2.5), with $g(t)$ replaced by 1, we find that

$$\overline{F}(y,s) = \frac{I(s,y)}{4(1-\frac{1}{2}s^{\frac{1}{2}})}\sum_{n=0}^{\infty}\left[\frac{1+\frac{1}{2}s^{\frac{1}{2}}}{1-\frac{1}{2}s^{\frac{1}{2}}}\right]^n e^{-ns^{\frac{1}{2}}},$$

where

$$\overline{F} = \int_0^\infty e^{-s\tau}\,F(y,\tau)\,d\tau$$

and

$$I(s,y) = \frac{1}{s}[1-s^{\frac{1}{2}}y(1+e^{-s^{\frac{1}{2}}})-e^{-s^{\frac{1}{2}}y}+e^{-s^{\frac{1}{2}}(1-y)}-e^{-s^{\frac{1}{2}}}].$$

For small τ

$$\overline{F}(y,s) = \frac{1}{s}\frac{[1-s^{\frac{1}{2}}y-e^{-s^{\frac{1}{2}}y}+e^{-s^{\frac{1}{2}}(1-y)}]}{4(1-\frac{1}{2}s^{\frac{1}{2}})}+O\left(\frac{e^{-s^{\frac{1}{2}}}}{s}\right),$$

so that

$$F(y,\tau) = \frac{1}{4}\left[1+2e^{4\tau}(y-\tfrac{1}{2})\operatorname{erfc}(-2\tau^{\frac{1}{2}})+\operatorname{erfc}\frac{(1-y)}{2\tau^{\frac{1}{2}}}-\operatorname{erfc}\frac{y}{2\tau^{\frac{1}{2}}}\right.$$

$$\left.+e^{-2y+4\tau}\operatorname{erfc}\left(\frac{y}{2\tau^{\frac{1}{2}}}-2\tau^{\frac{1}{2}}\right)-e^{-2(1-y)+4\tau}\operatorname{erfc}\left(\frac{(1-y)}{2\tau^{\frac{1}{2}}}-2\tau^{\frac{1}{2}}\right)\right]+O(\tau^{\frac{1}{2}}e^{-1/4\tau}). \tag{3.4}$$

A corresponding expression can be derived for the radial pressure gradient; this result may be employed to estimate the accuracy of the regular perturbation solution close to the start of the motion.

4. Numerical solution of equation (2.4)

When the upper surface is in motion with constant velocity the nth-order regular perturbation solution to (3.1) is of the form

$$F = \sum_{i=0}^{n} [Re\,(1-t)]^i f_i\left(\frac{z}{1-t}\right),$$

where the universal functions f_i are such that for $0 \leqslant i \leqslant 5$ $|f_{i+1}/f_i| = O(10^{-2})$ (MacDonald 1977). Thus for $Re = O(10^2)$ or larger and $(1-t)$ not small F will change appreciably as the order of the approximation changes. To test the accuracy of the perturbation solution over a wide range of Reynolds numbers and to obtain reliable information on the flow characteristics over this range a numerical solution of the governing nonlinear equation is necessary.

4.1. *Starting the solution*

Rapid acceleration of the upper surface from a state of rest to a state of uniform motion will, for small t, result in the formation of layers of concentrated vorticity adjacent to the boundaries. As there is a limit to the fineness of the mesh which is employed in the numerical scheme there will always exist a time range, close to the start of motion, within which it is not possible to obtain numerical results which accurately approximate the fluid velocity in these layers. In the limiting case of impulsive motion of the upper surface the initial condition for the integration of (2.4) is that

$$F = \tfrac{1}{2}y, \quad \tau = 0^+, \quad y \neq 0, 1,$$

and indeed when (2.4) is integrated subject to this condition it is found that when t is small the resulting solution is inaccurate near the boundaries. An alternative to starting the solution at $\tau = 0$ is to start at $\tau = \tau_s$, where $0 < \tau_s \ll 1$, and to employ as initial condition that value of F (F_s say) which is predicted by equation (3.4); for $\tau \geqslant \tau_s$ and for sufficiently rapid acceleration this solution, which is known to be accurate for $Re\,\tau^{\frac{1}{2}} = o(1)$, differs little from that corresponding to finite acceleration. This is the procedure which we followed; thus (2.4) was integrated subject to the condition

$$F = F_s(y), \quad \tau = \tau_s, \tag{4.1}$$

where $Re\,\tau_s^{\frac{1}{2}} = o(1)$. Further details relevant to the initial motion are presented in §4.3.

4.2. *The numerical scheme*

We employ an implicit finite-difference scheme of the Crank–Nicolson type. On the y axis we select uniformly spaced mesh points at $y_i = ih$, $i = 0, 1, \ldots, (m+1)$, where $(m+1)h = 1$. F will be evaluated at the times $\tau_j = jk$, $j = 1, 2, \ldots, (n-1)$, $nk = 1/Re$, and we shall denote the value of F at the mesh point (y_i, τ_j) by F_i^j. Central difference formulae are employed for the approximation of the differences in the y direction and a forward difference formula is used for the derivative in the τ direction. For example, we find that

$$\left.\frac{\partial^4 F}{\partial y^4}\right|_i^j = \frac{F_{i+2}^j - 4F_{i+1}^j + 6F_i^j - 4F_{i-1}^j + F_{i-2}^j}{h^4} + O\left(h^2 \left.\frac{\partial^6 F}{\partial y^6}\right|_i^j\right),$$

$$\left.\frac{\partial^3 F}{\partial y^2\,\partial \tau}\right|_i^j = \frac{F_{i+1}^{(j+1)} - 2F_i^{(j+1)} + F_{i-1}^{(j+1)} - F_{i+1}^j + 2F_i^j - F_{i-1}^j}{kh^2} + O\left(h^2 \left.\frac{\partial^5 F}{\partial y^4\,\partial \tau}\right|_i^j\right).$$

The Crank–Nicolson approximation scheme is now applied to equation (2.4). Thus all terms of the equation, other than $F_{yy\tau}$ – which is approximated directly as given above – are replaced by the mean of their values on the jth and $(j+1)$th time rows. The result is m equations of the form

$$F_i^{(j+1)} \sum_{s=1}^{5} a_{si}^{(j+1)} F_{i+3-s}^{(j+1)} + \sum_{s=1}^{5} b_{si}^{(j+1)} F_{i+3-s}^{(j+1)} = c_i^{(j+1)}, \quad i = 1, 2, \ldots, m, \tag{4.2}$$

where the coefficients a_{si}^{j+1}, b_{si}^{j+1}, c_i^{j+1} are defined in terms of $h, k, Re, \frac{1}{2}(\tau_j + \tau_{j+1})$ and the values of $F_{-1}^j, F_0^j, \ldots, F_{m+2}^j$ (the values F_{-1}^j and F_{m+2}^j follow from the boundary conditions $F_y = 0, y = 0, y = 1$). We select $10h = 1/2^l$, where l is an integer so that $(10 \times 2^l - 1)$ nonlinear algebraic equations must be solved. The boundary conditions (2.5) give, for $j = 0, 1, \ldots, (n-1)$,

$$F_0^j = 0, \quad F_{m+1}^j = \tfrac{1}{2}, \quad F_{-1}^j = F_1^j, \quad F_{m+2}^j = F_m^j.$$

The algebraic equations were solved by use of the Newton–Raphson iterative technique.

4.3. *Computational details*

The calculations were performed on a CDC 7600 machine which retained 15 significant figures. The range of Reynolds numbers studied was $0{\cdot}5 \leqslant Re \leqslant 96$ and results were compiled for τ in the range $\tau_s < \tau < 1/Re$. (τ_s varied with Re, being, for example, 0·02 at $Re = 0{\cdot}5$ and 0·0005 at $Re = 20$). For fixed Re, accuracy was checked by comparing the results for two consecutive l values – the range of l varying from 1 at the lower Reynolds numbers to 3 at the higher. For $l = 1$ k was selected to be 0·000625 so that the stability parameter k/h^2 was 0·25. For $l = 2$, k was taken to be 0·000078125 and the stability parameter was 0·125. The initial condition (4.1) was checked by comparing the numerical value of F at small τ with the value predicted by equation (3.4).

The radial pressure gradient is specified by the equation which results when (2.1) is transformed according to (2.3). For small values of τ, however, the finite-difference representation of $(1/r)\,\partial p/\partial r$ as obtained from this equation is unsatisfactory owing to inaccurate finite-difference approximations to derivatives which occur in the equation. It is preferable, therefore, to note that since $(1/r)\,\partial p/\partial r$ is independent of y the transformed version of equation (2.1) may be multiplied by $y(1-y)$ and integrated over the y range to yield the equation

$$\frac{h^2}{6r}\frac{\partial p}{\partial r} = \frac{-g}{4}\left\{\frac{4}{Re\,h} + g\right\} - 3\int_0^1 y(1-y)\,F_y^2\,dy - 2\int_0^1 F^2\,dy$$
$$+ g\left\{\int_0^1 (1-2y)\,F\,dy - 2\int_0^1 (1-3y)\,F\,dy\right\} + \frac{h}{Re}\frac{d}{d\tau}\int_0^1 (1-2y)\,F\,dy, \tag{4.3}$$

in which $h \equiv h(Re\,\tau)$ and $g \equiv g(Re\,\tau)$.

5. Results and discussion

We discuss the case of an upper surface which is rapidly accelerated from a state of rest to a state of uniform motion, the direction of motion being towards a lower surface which is at rest. Close to the start of the motion the vorticity layers adjacent to both boundaries are thin and the flow field is largely inviscid, the inviscid velocity distribution being specified, when $g'(t)$ is not large, by

$$\frac{u}{r} \equiv \frac{\partial F}{\partial z} = \frac{1}{2}\frac{g(t)}{h(t)},$$

where $g(t)$ and $h(t)$ respectively denote the speed of the moving surface and its distance of separation from the lower surface.

Rapid acceleration of the upper surface results in rapid acceleration of the fluid, this being achieved by the radial pressure gradient, which does work to overcome fluid inertia and frictional resistance; indeed, since $\partial p/\partial r$ is independent of z, the radial pressure gradient can be specified by the equation

$$-\frac{1}{r}\frac{\partial p}{\partial r} = \frac{1}{h(t)}\left\{3\int_0^{h(t)}\left(\frac{u}{r}\right)^2 dz + \tfrac{1}{2}g'(t) + \frac{2}{Re}\frac{\partial}{\partial z}\left(\frac{u}{r}\right)\bigg|_{z=0}\right\}, \tag{5.1}$$

which results when the radial momentum equation is integrated with respect to z.

It is of interest to examine the case where

$$g(t) = \operatorname{erfc}\alpha/t^{\frac{1}{2}}, \quad 0 < \alpha \ll 1. \tag{5.2}$$

In this case an order-of-magnitude analysis performed on equation (5.1) shows that for $\alpha/t^{\frac{1}{2}} = O(1)$ or larger the dominant term on the right-hand side of (5.1) is the inertial term $\frac{1}{2}g'(t)/h(t)$ which is $O(\alpha e^{-\alpha^2/t}/t^{\frac{3}{2}})$. On the other hand, for $\alpha/t^{\frac{1}{2}} = o(1)$

$$g'(t)\bigg/\frac{1}{Re}\frac{\partial}{\partial z}\left(\frac{u}{r}\right)\bigg|_{z=0} = O\left(\frac{\alpha}{t^{\frac{1}{2}}}\left(\frac{Re}{t}\right)^{\frac{1}{2}}\right),$$

i.e. when the diffusion length $(t/Re)^{\frac{1}{2}}$ is $O(\alpha/t^{\frac{1}{2}})$ the contribution due to frictional resistance is equal in importance to that due to fluid inertia (but when $(t/Re)^{\frac{1}{2}} = o(\alpha/t^{\frac{1}{2}})$, i.e. when $Re \gg 1$, frictional resistance is outweighed by inertial resistance). Indeed, as the surfaces approach one another inertial resistance becomes negligible in comparison to frictional resistance since the vorticity layers adjoining the surfaces will merge and the stream function $F(z,t)$ will tend to the classical lubrication value $F_0(z,t)$ (the manner in which $F \to F_0$ is, of course, dependent on Re). In such circumstances, replacement of u/r by $\partial F_0/\partial z$ in (5.1) shows that

$$\frac{\text{inertial resistance}}{\text{frictional resistance}} = O(Re\,(1-t)).$$

For given Re, we may thus distinguish three distinct phases in the radial pressure gradient development: in phase one $-(1/r)\,\partial p/\partial r$ rapidly increases in value and resistance to motion is due primarily to fluid inertia; in phase two $-(1/r)\,\partial p/\partial r$ rapidly decreases in value, the rate of change of fluid inertia is much reduced and the resistance due to friction assumes importance; in phase three $-(1/r)\,\partial p/\partial r$ again increases in value (tending to infinity as $t \to 1$) and the resistance due to friction outweighs that due to inertia. Frictional resistance will decrease with increase of Re, whereas inertial

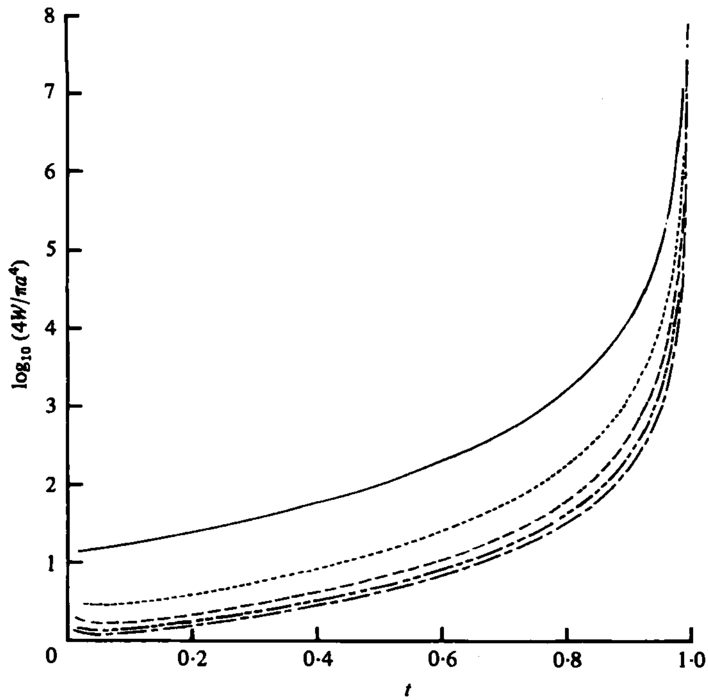

FIGURE 2. Normal force, or load, on a disk of radius a as a function of time t. ——, $Re = 0{\cdot}5$; -----, $Re = 5{\cdot}0$; – – –, $Re = 20{\cdot}0$; —·—; $Re = 48{\cdot}0$; ——, $Re = 96{\cdot}0$.

resistance will not; thus the minimum value of $-(1/r)\,\partial p/\partial r$ will decrease with increase of Re. The time, t_m, at which the minimum value of $-(1/r)\,\partial p/\partial r$ occurs is dependent on Re but is always close to the start of motion, when frictional resistance is not dominant. Our numerical results indicate that there is a finite, non-zero, Reynolds number for which t_m is a maximum but the prediction of this maximum value and the Reynolds number at which it occurs is not a straightforward numerical task. The best estimate which could be obtained was that the maximum value of t_m is approximately 0·058 and that this occurs at a Reynolds number of about 11·8 (note that $(Re\,t)^{\frac{1}{2}} = 0{\cdot}83$ so that the results obtained from the linearized analysis based on equation (3.3) are not reliable).

In the case of an upper surface in the shape of a disk of radius a, the normal force on the disk may be evaluated by use of equation (2.7). It is of interest to compare the results obtained when the radial pressure gradient in (2.7) is obtained from: (i) the numerical solution and (ii) the first-order regular perturbation solution. For Reynolds numbers of 0·5, 5·0, 20·0, 48·0, 96·0 and for values of t in the range $0{\cdot}1 \leqslant t \leqslant 0{\cdot}96$ this comparison is made in table 1.† The table shows that for Reynolds numbers up to 20 the agreement for values of t in the range $0{\cdot}1 \leqslant t \leqslant 0{\cdot}96$ is very good. For $Re = 48$ the agreement is acceptable, being to within ten per cent but by $Re = 96$ a discrepancy of almost fifteen per cent can be observed at the lower values of t. Table 2 compares the two solutions for $0{\cdot}0175 \leqslant t \leqslant 0{\cdot}16$. Within this range of t the classical lubrication

† The results quoted in tables 1 and 2 and figures 2–5 are those relevant to the case of impulsive motion.

Re ...	0·5		5·0		20·0		48·0		96·0	
t	$-P$	$-N$	$-P$	$-N$	$-P$	$-N$	$-P$	$-N$	$-P$	$-N$
0·10	17·7836	17·7669	2·96884	3·03998	1·73427	1·76320	1·49422	1·41734	1·40848	1·25763
0·20	25·1116	25·0909	4·01786	3·98358	2·26004	2·21525	1·91825	1·76450	1·79618	1·56350
0·36	48·3922	48·3608	7·19343	7·12826	3·76020	3·72925	3·09263	2·84966	2·85421	2·48728
0·66	314·581	314·476	39·7997	39·6595	16·9012	16·5895	12·4487	111·7674	10·8586	9·71516
0·90	12107·1	12105·0	1307·14	1311·05	407·143	405·446	232·143	225·512	169·643	158·120
0·96	188170	188163	19419·6	19411·3	5357·14	5354·00	2622·77	2596·32	1646·20	1577·26

TABLE 1. The radial pressure gradient $(1/r)\,\partial p/\partial r$ for $0 < t < 1$ as evaluated from: (i) the first-order perturbation solution (P) and (ii) the numerical solution (N).

Re ...	0·5		5·0		20·0		48·0		96·0	
t	$-P$	$-N$	$-P$	$-N$	$-P$	$-N$	$-P$	$-N$	$-P$	$-N$
0·0175	13·7626	14·1954	2·37521	3·24624	1·42625	1·91477	1·24173	1·46195	1·17584	1·27210
0·025	14·0740	14·1782	2·42177	3·06924	1·45075	1·80700	1·26194	1·40638	1·19451	1·22386
0·035	14·5042	14·5117	2·48592	2·92876	1·48440	1·73360	1·28966	1·36746	1·22011	1·19608
0·07	16·1575	16·1418	2·73066	2·90134	1·61176	1·69863	1·39419	1·36270	1·31649	1·20437
0·16	21·7646	21·7457	3·54308	3·53156	2·02462	1·99874	1·72936	1·60023	1·62391	1·42073

TABLE 2. The radial pressure gradient $(1/r)\,\partial p/\partial r$ for $0 < t \ll 1$ as evaluated from: (i) the first-order perturbation solution (P) and (ii) the numerical solution (N).

Re ...	0·5		5·0		20·0		96·0	
t	$-L$	$-N$	$-L$	$-N$	$-L$	$-N$	$-L$	$-N$
0·035	14·5320	14·5117	2·99042	2·92876	1·73439	1·73360	1·18249	1·19608
0·100	17·7836	17·7669	3·08812	3·03998	1·76786	1·76320	1·25540	1·25763
0·360	48·3921	48·3608	7·19512	7·12826	3·68872	3·72925	2·48851	2·48728
0·660	314·581	314·476	39·7996	39·6595	16·7825	16·5895	9·72052	9·71516
0·900	12107·1	12105·0	1307·14	1311·05	406·703	405·446	158·141	158·120
0·960	188170	188163	19419·6	19411·3	5355·60	5354·00	1576·81	1577·26

TABLE 3. The radial pressure gradient $(1/r)\,\partial p/\partial r$ for $0 < t < 1$ as evaluated from: (i) the linearized analysis based on equation (3.3) (L) and (ii) the numerical solution (N).

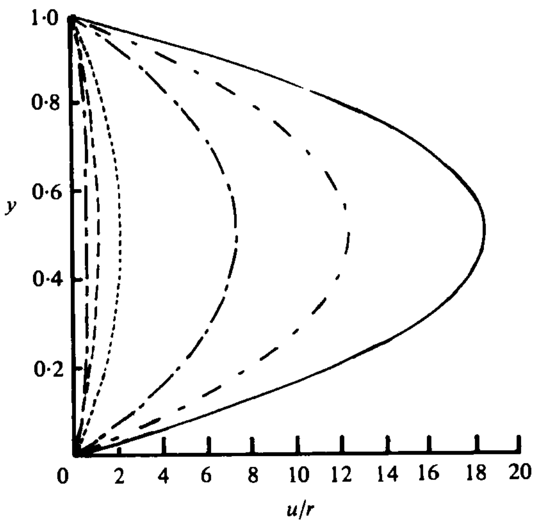

FIGURE 3. Radial velocity profile development with time t: $Re = 5{\cdot}0$. ———, $t = 0{\cdot}035$; ---, $t = 0{\cdot}36$; ·····, $t = 0{\cdot}66$; —·—·, $t = 0{\cdot}9$; —·—, $t = 0{\cdot}94$; ——, $t = 0{\cdot}96$.

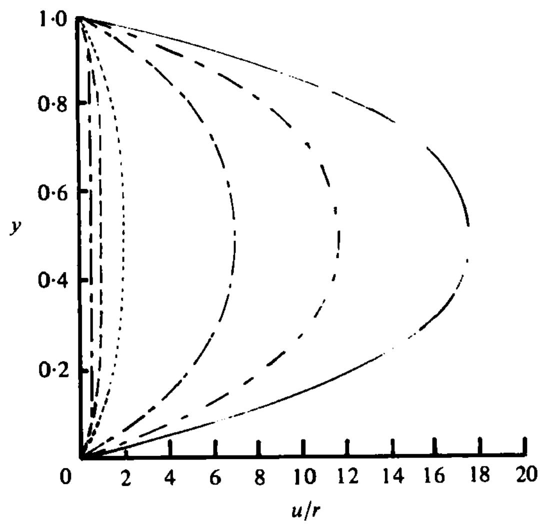

FIGURE 4. Radial velocity profile development with time t: $Re = 48{\cdot}0$. ———, $t = 0{\cdot}035$; ---, $t = 0{\cdot}36$; ·····, $t = 0{\cdot}66$; —·—·, $t = 0{\cdot}9$; —·— $t = 0{\cdot}94$; ——, $t = 0{\cdot}96$.

approximation to $(1/r)\,\partial p/\partial r$ is $-(6/Re)\,(1-t)^{-3}$, so that at $Re = 0{\cdot}5$ both sets of results do not differ appreciably from the classical lubrication result, whereas for $Re \geqslant 5{\cdot}0$ they do. For $Re = 96$ the percentage error in the quoted values of $(1/r)\,\partial p/\partial r$ decreases as t increases to 0·035 but thereafter it increases to reach 14·3 % at $t = 0{\cdot}16$. Figure 2 presents the load variation with t for a range of Reynolds numbers extending to $Re = 96$. The figure indicates that for values of t not too close to zero the load on the disk decreases with increase of Re. Expressed in dimensional terms, this result states that if V and H are held constant (so that the time scale H/V remains

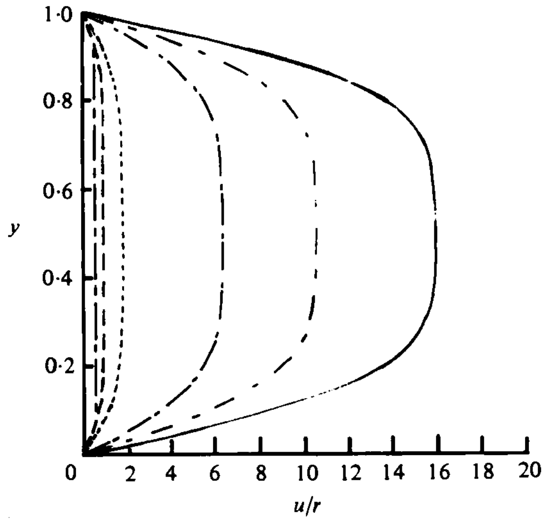

FIGURE 5. Radial velocity profile development with time t: $Re = 96{\cdot}0$. ———, $t = 0{\cdot}035$; ---, $t = 0{\cdot}36$; ·····, $t = 0{\cdot}66$; —·—·, $t = 0{\cdot}9$; —·—, $t = 0{\cdot}94$; ———, $t = 0{\cdot}96$.

constant) a decrease in kinematic viscosity will result in a decrease in the load on the disk.

It is worth while to compare the numerical solution for the radial pressure gradient with the result obtained from equation (4.3) and the solution to the linearized equation (3.3). For a range of Reynolds numbers extending to $Re = 96$, this comparison is made in table 3. The table demonstrates the remarkable agreement between the two approximate solutions.

An elementary calculation based on the diffusion length $(t/Re)^{\frac{1}{2}}$ shows that the vorticity layers adjacent to the boundaries will merge in a time of order t_m, where

$$t_m = 2/Re + 1 - 2(1+Re)^{\frac{1}{2}}/Re;$$

thus for $Re \ll 1$ $t_m \sim \frac{1}{4}Re + O(Re^2)$, whereas for $Re \gg 1$ $t_m \sim 1 - 2/Re^{\frac{1}{2}}$. The influence of Reynolds number on the rate at which vorticity diffuses from the boundaries can be gauged from figures 3, 4 and 5 which present radial velocity profiles for Reynolds numbers of 5·0, 48·0 and 96·0 respectively.

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